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GALERKIN METHODS FOR MISCIBLE
DISPLACEMENT PROBLEMS IN POROUS MEDIA

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GALERKIN METHODS FOR MISCIBLE DISPLACEMENT

PROBLEMS IN POROUS MEDIA

Richard E. Ewing[1] and Mary Fanett Wheeler[2]

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ABSTRACT

A priori error estimates for Galerkin methods for numerical approximation

of the coupled quasilinear system for $c = c(x,t)$ and $p = p(x,t)$ given by

$$\nabla \cdot [a(x,c)\{\nabla p - \gamma(x,c)\nabla z\}] = 0 \ ,$$

$$\nabla \cdot [b(x,c,\nabla p)\nabla c] - u(x,c,\nabla p) \cdot \nabla c = \phi(x) \frac{\partial c}{\partial t} \ ,$$

for $x \in \Omega$, $t \in (0,T]$, and appropriate Neumann boundary and initial condi-

tions are considered. Equations of this type arise in models for the miscible

displacement of one incompressible fluid by another in a porous medium.

Estimates for both continuous time and fully-discrete time Galerkin methods

are presented.

AMS (MOS) Subject Classifications: 65M15, 65N15, 65N30, 76.35
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Work Unit Number 7 (Numerical Analysis)

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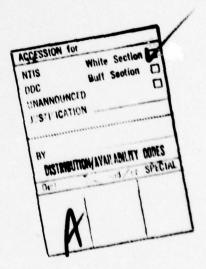
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SIGNIFICANCE AND EXPLANATION

Equations of the type stated in the Abstract arise, for example, in models for the miscible displacement of one incompressible fluid by another in a porous medium as in chemical flooding of oil wells to recover more of the oil. The variable $c = c(x,t)$ corresponds to the concentration of the fluid used for the flooding process while the variable $p = p(x,t)$ corresponds to the pressure at the location x in the medium at time t.

This paper gives error analysis and error estimates for certain numerical methods for approximate solution of the coupled quasilinear system of parabolic partial differential equations in the Abstract. Both continuous-time and fully-discrete time methods are presented and analyzed.

GALERKIN METHODS FOR MISCIBLE DISPLACEMENT PROBLEMS IN POROUS MEDIA

Richard E. Ewing[1] and Mary Fanett Wheeler[2]

1. Introduction

We consider the numerical approximation by Galerkin methods to a problem arising in the miscible displacement of one incompressible fluid by another in a porous medium. A set of equations [6] and [9] modeling the pressure $p(x,y,t)$ and the concentration $c(x,y,t)$ is given by

$$(1.1) \quad \frac{\partial}{\partial x}\left(\frac{k_x}{\mu(c)}\left(\frac{\partial p}{\partial x}-\gamma(c)\frac{\partial z}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(\frac{k_y}{\mu(c)}\left(\frac{\partial p}{\partial y}-\gamma(c)\frac{\partial z}{\partial y}\right)\right) \equiv -\frac{\partial}{\partial x}u_x - \frac{\partial}{\partial y}u_y = q(x,y)$$

and

$$\frac{\partial}{\partial x}\left(\alpha(\phi,D,c,\nabla p)\frac{\partial c}{\partial x}-u_x c\right) + \frac{\partial}{\partial y}\left(\delta(\phi,D,c,\nabla p)\frac{\partial c}{\partial y}-u_y c\right)$$

$$(1.2) \qquad + \frac{\partial}{\partial y}\left(\beta(\phi,D,c,\nabla p)\frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial x}\left(\beta(\phi,D,c,\nabla p)\frac{\partial c}{\partial y}\right)$$

$$= \phi(x,y)\frac{\partial c}{\partial t} + q(x,y)\hat{c}(x,y,t) ,$$

for $(x,y) \in \Omega$, $t \in J = (0,T]$, where the reservoir Ω is a bounded domain with boundary $\partial\Omega$. Here $k_x = k_x(x,y)$ and $k_y = k_y(x,y)$ are the permeabilities in the x and y directions respectively, $\mu = \mu(c)$ is the local viscosity of the fluid, $\gamma = \gamma(c)$ is its density, $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are the components of the dip angle, and q, the imposed total flow rate, is a linear combination of a finite number of Dirac measures, (i.e. sources and sinks). The function $\phi = \phi(x,y)$ is the porosity and $D = D(x,y)$ is the molecular diffusion level. The components of the dispersion coefficient, α, δ and β, are functions of porosity, diffusion, concentration, and the gradient of pressure. It is not uncommon in petroleum reservoir engineering for one to assume that $\beta = 0$ and $\alpha = \alpha(D)$ and $\delta = \delta(D)$ (see [9]). The function $\hat{c}$ is equal to the specified concentration at injection wells (sources) and to $c(x,y,t)$ for production wells (sinks). No flow

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conditions are assumed on the boundary of the reservoir and an initial concentration is given throughout the reservoir. For a general discussion of the physics of miscible displacement problems in reservoir engineering the reader is referred to Peaceman [7].

In the Galerkin procedure we shall use different subspaces of $H^1(\Omega)$ to approximate pressure and concentration. We note that, if $c(x,y,t) \equiv 1$, then (1.1) and (1.2) are identical. In this case we are forced to use the same approximating subspaces for pressure and concentration. We are thus led to replacing (1.2) by the non-divergence form

(1.3)
$$\frac{\partial}{\partial x}\left(\alpha \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(\delta \frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial y}\left(\beta \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial x}\left(\beta \frac{\partial c}{\partial y}\right) - u_x \frac{\partial c}{\partial x} - u_y \frac{\partial c}{\partial y}$$
$$= \phi \frac{\partial c}{\partial t} + q(x,y)(\hat{c} - c) ,$$

which is obtained by multiplying (1.1) by c and subtracting the result from (1.2). Settari, Price, and Dupont [9] presented, without analysis, a finite element method for solving (1.1)-(1.3) with $\beta \equiv 0$. For simplicity, we shall treat the coupled quasilinear system with dependent variables $c = c(x,t)$ and $p = p(x,t)$ given by

(1.4)
$$\nabla \cdot [a(x,c)(\nabla p - \gamma(x,c)\nabla z)] = 0 ,$$

(1.5)
$$\nabla \cdot [b(x,c,\nabla p)\nabla c] - u(x,c,\nabla p) \cdot \nabla c = \phi(x) \frac{\partial c}{\partial t} ,$$

for $x \in \Omega$, $t \in J \equiv (0,T]$, where Ω is a bounded domain in $\mathbb{R}^d$, $d = 2$, with boundary $\partial\Omega$ and $u(x,c,\nabla p) = -a(x,c)(\nabla p - \gamma(x,c)\nabla z)$ is a vector in $\mathbb{R}^2$. The analysis for (1.4)-(1.5) easily extends to the case (1.1) and (1.3) where mixed $x - y$ derivatives are present. We assume that the following boundary and initial conditions hold:

(1.6)
$$a(x,c)\left\{\frac{\partial p}{\partial \nu} - \gamma(x,c) \frac{\partial z}{\partial \nu}\right\} = 0, \qquad x \in \partial\Omega, \quad t \in J ,$$

(1.7)
$$b(x,c,\nabla p) \frac{\partial c}{\partial \nu} = 0 , \qquad x \in \partial\Omega, \quad t \in J ,$$

(1.8)
$$c(x,0) = c_0(x) , \quad x \in \Omega ,$$

where $\frac{\partial f}{\partial \nu}$ is the normal derivative of f on the boundary of Ω. We note that (1.4)-(1.8) will define $p(x,t)$ only to within an arbitrary constant. We shall

normalize p by the condition that

(1.9)
$$\frac{1}{|\Omega|} \int_{\Omega} p(x,t)dx = 1, \qquad t \in J ,$$

where $|\Omega|$ is the measure of Ω . We note that for (1.4) and (1.5), we have chosen
$q \equiv 0$ in (1.1) and (1.3). If q is assumed to be smooth, which corresponds to smoothly
distributed sources and sinks, the analysis follows with few changes.

The paper consists of three additional sections. In Section 2, the continuous
time and discrete time Galerkin procedures are formulated. Assumptions on the coefficients
corresponding to the physical problem and smoothness assumptions on the solution are
given. Methods are presented which allow the approximations for concentration and
pressure to lie in different subspaces of $H^1(\Omega)$. In Section 3, a-priori error estimates
for the continuous time approximations are given. Optimal L^2 rates of convergence are
established for the case where the dispersion coefficient depends only upon molecular
diffusion using subspaces of the same order of approximation. In the case that the
dispersion coefficient also depends upon the Darcy velocity, we obtain an optimal L^2
rate using a subspace of one higher order for the pressure than for the concentration.
In Section 4, these results are extended to the discrete time case.

2. The Finite Element Methods and a Précis of the Results

Let $(u,v) = \int_\Omega uv \, dx$ and $\|u\|^2 = (u,u)$. Let $W_s^k(\Omega)$ be the Sobolev space on Ω with norm

$$\|\phi\|_{W_s^k} = \left(\sum_{|\alpha| \le k} \left\| \frac{\partial^\alpha \phi}{\partial x^\alpha} \right\|_{L^s(\Omega)}^s \right)^{\frac{1}{s}} .$$

with the usual modification for $s = \infty$. When $s = 2$, let $\|\phi\|_{W_2^k} = \|\phi\|_{H^k} \equiv \|\phi\|_k$. If $\nabla f = (f_1, f_2)$ write $\|\nabla f\|_{W_s^k}$ in place of $(\|f_1\|_{W_s^k}^s + \|f_2\|_{W_s^k}^s)^{\frac{1}{s}}$. Also $H^s(\partial\Omega)$ will denote the usual Sobolev space on $\partial\Omega$.

Let $\{M_h\}$ be a family of finite-dimensional subspaces of $H^1(\Omega)$ with the following property:

> For $p = 2$ or $p = \infty$, there exist an integer $r \ge 2$ and a constant K_0 such that, for $1 \le q \le r$ and $\phi \in W_p^q(\Omega)$;

$$(2.1) \qquad \inf_{\chi \in M_h} \{\|\phi - \chi\|_{W_p^0} + h\|\phi - \chi\|_{W_p^1}\} \le K_0 \|\phi\|_{W_p^q} h^q .$$

Similarly, we define a family of finite-dimensional subspaces of $H^1(\Omega)$ called $\{N_h\}$ which satisfies the same property as $\{M_h\}$ with r replaced by s. We also assume that the families $\{M_h\}$ and $\{N_h\}$ satisfy the following so-called "inverse hypotheses":

$$(2.2) \qquad \text{a)} \quad \|\phi\|_{L^\infty(\Omega)} \le K_0 h^{-\frac{d}{2}} \|\phi\| = K_0 h^{-1} \|\phi\| , \quad \text{and}$$

$$\text{b)} \quad \|\nabla\phi\|_{L^\infty(\Omega)} \le K_0 h^{-1} \|\nabla\phi\| .$$

Restrict Ω as follows (with S denoting the collection of restrictions):

(S)   1) Ω is H^2-regular, i.e., if

$$-\Delta v + \theta v = \zeta, \qquad x \in \Omega, \quad \theta = 0 \text{ or } 1 ,$$

$$\frac{\partial v}{\partial \nu} = \eta, \qquad x \in \partial\Omega ,$$

and $(\zeta, 1) + \int_{\partial\Omega} \eta \, ds = 0$ if $\theta = 0$,

then

$$\|v\|_2 \le K(\Omega)(\|\zeta\| + \|\eta\|_{H^{1/2}(\partial\Omega)}) ;$$

2) $\partial\Omega$ is Lipschitz.

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Let

$$b \equiv b(x,c,\nabla p) \equiv b(x,c,p_x,p_y)$$

and

$$u \equiv u(x,c,\nabla p) = (u_1(x,c,p_x),\ u_2(x,c,p_y))\ .$$

For some $\hat{\epsilon} > 0$ restrict the variable q_1 to lie between

$$-\hat{\epsilon} \le q_1 \le 1 + \hat{\epsilon}\ .$$

Assume the following regularity for a, γ, b, u, and ϕ:

(Q):   1)   There exist constants a_*, b_*, ϕ_* and K_1 such that

a) $0 < a_* \le a(x,q_1) \le K_1$,

b) $|\gamma(x,q_1)| \le K_1$,

c) $0 < \phi_* \le \phi(x) \le K_1$,

(2.3)

d) $|\nabla z(x)| \le K_1$,

e) $0 < b_* \le b(x,\tilde{q},q_2,q_3)$, $\tilde{q},q_2,q_3 \in \mathbb{R}$,

f) $|u_i(x,q_1,q_4)| \le K_1(1 + |q_4|)$, $i = 1,2, q_4 \in \mathbb{R}$,

g) $|b(x,c,\nabla p)| \le K_1$.

2)   There exists a constant M such that for (c,p) the solution to (1.4)-(1.8) and for $(q_2,q_3) \in \mathbb{R}^2$, $i = 1,2$,

$$\left|\frac{\partial a}{\partial c}(x,q_1)\right| + \left|\frac{\partial \gamma}{\partial c}(x,q_1)\right| + \left|\frac{\partial^2 b}{\partial c^2}(x,c,\nabla p)\right| + \left|\frac{\partial u_i}{\partial q_1}(x,q_1,\nabla p)\right|$$

(2.4)

$$+ \left|\frac{\partial u_i}{\partial q_2}(x,q_1,q_2)\right| + \left|\frac{\partial^2 u_i}{\partial x_i \partial q_2}(x,c)\right| + \left|\frac{\partial^2 u_i}{\partial c^2}(x,c,\nabla p)\right| + \left|\frac{\partial^2 u_i}{\partial c \partial q_2}(x,c)\right|$$

$$+ \left|\frac{\partial b}{\partial q_1}(x,q_1,p_x,p_y)\right| + \left|\frac{\partial b}{\partial q_2}(x,q_1,q_2,q_3)\right| + \left|\frac{\partial b}{\partial q_3}(x,q_1,q_2,q_3)\right| \le M\ .$$

The assumptions made on the coefficients in (Q) are physically motivated. In the general miscible displacement formulation (1.1)-(1.3)

$$\alpha = \phi\left\{D + \alpha_\ell\frac{u_x^2}{\||u\||} + \alpha_t\frac{u_y^2}{\||u\||}\right\}\ ,$$

$$\beta = \phi\left\{D + (\alpha_\ell - \alpha_t)\frac{u_x u_y}{\||u\||}\right\}\ , \quad\text{and}$$

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$$\delta = \phi \left\{ D + a_\ell \frac{u_y^2}{|||u|||} + a_t \frac{u_x^2}{|||u|||} \right\} ,$$

where $|||u||| = \sqrt{u_x^2 + u_y^2}$ and a_ℓ and a_t are nonnegative constants (see [6] and [9]). One can show that a, β, and δ satisfy the conditions assumed for b, that $k_x/\mu(c)$ and $k_y/\mu(c)$ satisfy those for a, and that u_x and u_y satisfy assumptions made on the u_i.

Let

$$\| \phi \|_{L^q((a,b),X)} = \left\| \| \phi(\cdot,t) \|_X \right\|_{L^q(a,b)} , \qquad 1 \le q \le \infty .$$

Let (p,c), the solution of $(1.4)-(1.8)$, satisfy the following regularity assumptions:

(R) : a) $\|c\|_{L^\infty(J;H^r)} + \left\|\frac{\partial c}{\partial t}\right\|_{L^2(J;H^r)} + \|p\|_{L^\infty(J;H^s)} \le K_2 .$

(2.5) b) $\|c\|_{L^\infty(J;H^{2+\epsilon})} + \left\|\frac{\partial c}{\partial t}\right\|_{L^\infty(J;W_3^1)} + \left\|\frac{\partial^2 c}{\partial t^2}\right\|_{L^2(J;L^2)} + \left\|\frac{\partial p}{\partial t}\right\|_{L^\infty(J;H^1)} \le K_2 ,$

for some $\epsilon > 0$.

The analysis proceeds, following Wheeler [10], using a pair of auxiliary elliptic problems. Let $\tilde{p} \in N_h$ be the elliptic projection of p into N_h defined by

(2.6) $\qquad (a(\cdot,c(\cdot,t))\nabla\tilde{p},\nabla v) = (a(\cdot,c(\cdot,t))\nabla p,\nabla v), \qquad v \in N_h ,$

for each $t \in J$, where

(2.7) $\qquad \dfrac{1}{|\Omega|} \displaystyle\int_\Omega \tilde{p}(x,t)dx = 1, \quad \text{for each} \quad t \in J ,$

and where (p,c) is the solution of $(1.4)-(1.8)$. The restrictions (S) imply the following result.

Lemma 2.1. There exists $K_3 = K_3(\Omega, a_\epsilon, K_0, K_1)$ such that

(2.8) $\qquad \|\tilde{p} - p\| + h\|\nabla(\tilde{p} - p)\| \le K_3 h^s \|p\|_s .$

Let $\lambda > 0$ be chosen sufficiently large that the bilinear form

$$B(\psi,\chi) = (b(c,\nabla p)\nabla\psi,\nabla\chi) + (u(c,\nabla p)\cdot\nabla\psi,\chi) + \lambda(\psi,\chi)$$

is coercive over $H^1(\Omega)$. Let $\tilde{c} \in M_h$ be the elliptic projection of c into M_h, defined by

(2.9) $\qquad B(\tilde{c},w) = B(c,w) = -(\phi \frac{\partial c}{\partial t}, w) + \lambda(c,w), \qquad w \in M_h .$

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for each $t \in J$. Then, as in [10] and [3], we can obtain the following lemma.

Lemma 2.2. There exists $K_4 = K_4(\Omega, b_*, \lambda, K_0, K_1, K_2, M)$ such that

a) $\left\| \tilde{c} - c \right\|_{L^2(J;L^2)} + \left\| \dfrac{\partial(\tilde{c} - c)}{\partial t} \right\|_{L^2(J;L^2)} + h\|\tilde{c} - c\|_{L^2(J;H^1)}$

$$(2.10) \qquad \le K_4 h^r \left\{ \|c\|_{L^2(J;H^r)} + \left\| \frac{\partial c}{\partial t} \right\|_{L^2(J;H^r)} \right\},$$

b) $\|\tilde{c} - c\|_{L^\infty(J;L^2)} \le K_4 h^r \|c\|_{L^\infty(J;H^r)}$.

Assume that there exists a constant K_5 such that

$$(2.11) \qquad \|\nabla \tilde{p}\|_{L^\infty(J;L^\infty)} + \|\nabla \tilde{c}\|_{L^\infty(J;L^\infty)} \le K_5 .$$

See [2] and [10] for some sufficient conditions for these assumptions.

We first consider continuous time approximations of (p,c). Denote the approxima-
tion of p by $P : [0,T] \to N_h$ and the approximation of c by $C : [0,T] \to M_h$,
where (P,C) is defined by the relations: (we suppress the dependence of the coefffi-
cients on x)

$$(2.12) \qquad (a(C)\nabla P, \nabla v) = (a(C)\gamma(C)\nabla z, \nabla v), \qquad v \in N_h ,$$

and

$$(2.13) \qquad (b(C,\nabla P)\nabla C, \nabla w) + (u(C,\nabla P)\cdot\nabla C, w) + (\phi \frac{\partial c}{\partial t}, w) = 0, \qquad w \in M_h ,$$

with

$$(2.14) \qquad C(x,0) = \tilde{c}(x,0) ,$$

where $\tilde{c}$ is the elliptic projection of c defined in (2.9). In Section 3, we shall
obtain a priori estimates for $c - C$ and for $\nabla(p - P)$.

Next, we define a fully discrete, $O(\Delta t)$-correct method for approximating (p,c)
based on backward differencing in time. Let $\Delta t > 0$, $N = T/\Delta t \in \mathbb{Z}$, and $t^\sigma = \sigma \Delta t$,
$\sigma \in \mathbb{N}$. Also, let $\psi^n = \psi^n(x) = \psi(x,t^n)$ and $d_t\psi^n = (\psi^{n+1} - \psi^n)/\Delta t$. Denote the
approximation of p by $W : \{0 = t^0, t^1, \ldots, t^N = T\} \to N_h$ and the approximation of c

by $Z : \{t^0, t^1, \ldots, t^N\} \to M_h$. Assuming that W^n and Z^n are known, we determine W^{n+1} and Z^{n+1} as follows:

$$(2.15) \quad (\phi d_t Z^n, \psi) + (b(Z^n, \nabla W^n) \nabla Z^{n+1}, \nabla \psi) + (u(Z^n, \nabla W^n) \cdot \nabla Z^{n+j}, \psi) = 0, \quad \psi \in M_h,$$

and

$$(2.16) \quad (a(Z^{n+1}) \nabla W^{n+1}, \nabla y) = (a(Z^{n+1}) \gamma(Z^{n+1}) \nabla z, \nabla y), \quad y \in N_h,$$

where $j = 0$ or 1. We note that the coefficient matrix arising from the algebraic system (2.15) with $j = 0$ is symmetric. However, in many problems the transport term is large and it may be numerically advantageous to use (2.15) with $j = 1$ even though the coefficient matrix is no longer symmetric. Both discretizations are $O(\Delta t)$ in time. In (2.16) no explicit time discretization error is made. A priori error estimates will be presented for $c^n - Z^n$ and $\nabla(P^n - W^n)$ in Section 4.

Let $f^{n+1/2} \equiv (f^{n+1} + f^n)/2$. If (2.15) is replaced by

$$(2.17) \quad (\phi d_t \tilde{Z}^n, \psi) + (b(E\tilde{Z}^n, \nabla EW^n) \nabla \tilde{Z}^{n+\frac{1}{2}}, \nabla \psi) + (u(E\tilde{Z}^n, \nabla EW^n) \cdot \nabla \tilde{Z}^{n+\frac{1}{2}}, \psi) = 0, \quad \psi \in M_h,$$

where $E\tilde{Z}^n = \frac{3}{2} \tilde{Z}^n - \frac{1}{2} \tilde{Z}^{n-1}$, an analysis similar to that given in Section 4 (see [8], [10], [2] and [5]) will yield a time discretization error of size $O((\Delta t)^2)$.

When estimating the actual physical quantities, we know that the pressure is much smoother in time than the concentration. Thus, in practice, one should not use (2.16) to redetermine ∇W^{n+1} at each time step. Instead one value of ∇W^{n+1} should be used in the coefficients of (2.15) for several time steps before a new pressure is approximated. One way to view this procedure is to have two different time steps for the different equations. An analysis of this method will follow in a more general paper.

3. A Priori Error Estimates for the Continuous Time Approximations

In this section, we develop a priori bounds for the errors $(c - C)$ and $\nabla(p - P)$. We shall see that if b depends on ∇p then for the best rates of convergence one should use subspaces of piecewise polynomials of different degrees for $\{M_h\}$ and $\{N_h\}$. We first obtain an estimate for $P - \tilde{p}$ where P is defined by (2.12)-(2.14) and $\tilde{p}$ is defined by (2.6)-(2.7).

We assume that for some $\varepsilon > 0$, $\varepsilon \le \hat{\varepsilon}$, (see the statement preceding Q),

$$(3.0) \qquad -\varepsilon \le C \le 1 + \varepsilon .$$

We shall prove that

$$\| c - C \| = O(h^r + h^{s-1}) ,$$

which yields

$$(3.1) \qquad \| c - C \|_{L^\infty} \le K(h^{r-1} + h^{s-2}) .$$

Since $0 \le c \le 1$, one can deduce from (3.1) with the restriction $r \ge 2$ and $s \ge 3$ that (3.0) holds for small h.

Lemma 3.1. There exists a positive constant $K_6 = K_6(a_*, K_1, K_5, M)$ such that

$$\| \nabla(P - \tilde{p}) \| \le K_6 \| C - c \| , \qquad t \in J .$$

Proof. Subtract (2.6) from (2.12) and use (1.4) and (1.6) to obtain

$$(3.2) \quad (a(C)\nabla(P - \tilde{p}), \nabla v) = ([a(c) - a(C)]\nabla\tilde{p}, \nabla v) + ([a(C)\gamma(C) - a(c)\gamma(c)]\nabla z, \nabla v), \qquad v \in N_h .$$

If $v = P - \tilde{p} \in N_h$, then

$$a_* \| \nabla(P - \tilde{p}) \|^2 \le |(a(C)\nabla(P - \tilde{p}), \nabla(P - \tilde{p}))| \le K_7 \| C - c \| \, \| \nabla(P - \tilde{p}) \|$$

$$(3.3)$$

$$\le \frac{a_*}{2} \| \nabla(P - \tilde{p}) \|^2 + \frac{K_7^2}{2a_*} \| C - c \|^2, \qquad t \in J ,$$

where K_7 depends upon uniform bounds for $|\frac{\partial a}{\partial c}|$, $|\frac{\partial \gamma}{\partial c}|$, $|\nabla z|$, and $\| \nabla\tilde{p} \|_{L^\infty(J; L^\infty)}$. The result of the lemma follows directly from (3.3).

We next compare C and $\tilde{c}$, defined in (2.12)-(2.14) and (2.9), respectively. We shall make an additional assumption on P defined in (2.12)-(2.14). Assume that there exists a positive constant K^* such that

$$(3.4) \qquad \| \nabla P \|_{L^\infty(J; L^\infty)} \le K^* .$$

Without loss of generality, assume $K^* \ge 2K_5$.

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<u>Theorem 3.2.</u> There exists $K_8 = K_8(\lambda, b_*, \phi_*, M; K_i, i \leq 6; K^*)$ such that

(3.5)
$$\|C - \tilde{c}\|_{L^\infty(J;L^2)} + \|\nabla(C - \tilde{c})\|_{L^2(J;L^2)} \leq K_8(h^r + h^{s-1}) .$$

If s and r from Lemmas 2.1 and 2.2 satisfy $s \geq 3$ and $r \geq 2$ and h is taken sufficiently small, then

(3.6)
$$\|\nabla P\|_{L^\infty(J;L^\infty)} \leq 2K_5 \leq K^* $$

and a K_8 above can be chosen which is independent of K^* .

<u>Proof.</u> Subtract (2.9) from (2.13) to obtain

$$(b(C, \nabla P)\nabla(C - \tilde{c}), \nabla w) + \left(\phi\left(\frac{\partial C}{\partial t} - \frac{\partial \tilde{c}}{\partial t}\right), w\right) = ([b(c, \nabla p) - b(C, \nabla P)]\nabla\tilde{c}, \nabla w)$$

(3.7)
$$+ \left(\phi\left(\frac{\partial c}{\partial t} - \frac{\partial \tilde{c}}{\partial t}\right), w\right) - (u(C, \nabla P) \cdot \nabla(C - \tilde{c}), w)$$

$$+ ([u(c, \nabla p) - u(C, \nabla P)] \cdot \nabla\tilde{c}, w) + \lambda(\tilde{c} - c, w), \qquad w \in M_h .$$

Next, letting $w = C - \tilde{c} \in M_h$, integrating the left side of (3.7) from 0 to t and using (2.14), we obtain

$$\int_0^t (\phi \frac{\partial}{\partial\tau} (C - \tilde{c}), (C - \tilde{c}))d\tau + \int_0^t (b(C, \nabla P)\nabla(C - \tilde{c}), \nabla(C - \tilde{c}))d\tau$$

(3.8)
$$\geq \frac{\phi_*}{2} \|(C - \tilde{c})(t)\|^2 + b_* \int_0^t \|\nabla(C - \tilde{c})(\tau)\|^2 d\tau .$$

We next use (2.8), Lemma 2.2, Lemma 3.1 and Hölder's inequality to see that

$$\left|\int_0^t ([b(c, \nabla p) - b(C, \nabla P)]\nabla\tilde{c}, \nabla(C - \tilde{c}))d\tau\right|$$

$$\leq K_9\|\nabla\tilde{c}\|_{L^\infty(J;L^\infty)} \int_0^t (\|c - C\| + \|\nabla(p - P)\|)\|\nabla(C - \tilde{c})\|d\tau$$

$$\leq K_{10} \int_0^t (\|c - \tilde{c}\| + \|\tilde{c} - C\| + \|\nabla(p - \tilde{p})\|)\|\nabla(C - \tilde{c})\|d\tau$$

(3.9)
$$\leq \frac{b_*}{4} \int_0^t \|\nabla(C - \tilde{c})(\tau)\|^2 d\tau + K_{11} \int_0^t \|(C - \tilde{c})(\tau)\|^2 d\tau$$

$$+ K_{12} \int_0^t \{\|(c - \tilde{c})(\tau)\|^2 + \|\nabla(p - \tilde{p})(\tau)\|^2\}d\tau$$

$$< \frac{b_*}{4} \int_0^t \|\nabla(C - \tilde{c})(\tau)\|^2 d\tau + K_{11} \int_0^t \|(C - \tilde{c})(\tau)\|^2 d\tau + K_{13}\{h^{2r} + h^{2s-2}\} .$$

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Here K_{11} depends upon M, K_1, K_5, K_6, and b_*, while K_{13} depends upon M, K_1, K_2, K_3 K_4, K_5, K_6 and b_*. We can similarly obtain the bound

$$(3.10) \quad \left| \int_0^t \left([u(c,\nabla p) - u(C,\nabla P)] \cdot \nabla \tilde{c}, C - \tilde{c} \right) d\tau \right| \leq K_{14} \int_0^t \| (C - \tilde{c})(\tau) \|^2 d\tau + K_{15}\{h^{2r} + h^{2s-2}\} .$$

Next, from (2.3), we note that

$$(3.11) \quad \left| \int_0^t (u(C,\nabla P) \cdot \nabla (C - \tilde{c}), C - \tilde{c}) d\tau \right|$$
$$\leq K_{16}\{ \| \nabla P \|_{L^\infty(J;L^\infty)} + 1\}^2 \int_0^t \| (C - \tilde{c})(\tau) \|^2 d\tau + \frac{b_*}{8} \int_0^t \| \nabla (C - \tilde{c})(\tau) \|^2 d\tau .$$

Then using Lemma 2.2, we obtain

$$(3.12) \quad \left| \int_0^t \left[\phi\left(\frac{\partial c}{\partial \tau} - \frac{\partial \tilde{c}}{\partial \tau}\right), C - \tilde{c} \right] d\tau + \lambda \int_0^t (\tilde{c} - c, C - \tilde{c}) d\tau \right| \leq K_{17} \int_0^t \| (C - \tilde{c})(\tau) \|^2 d\tau + K_{17} h^{2r} .$$

Combining like terms in the estimates (3.8)-(3.12) and the assumption (3.4), we obtain, from (3.7),

$$(3.13) \quad \frac{\phi_*}{2} \| (C - \tilde{c})(t) \|^2 + \frac{b_*}{2} \int_0^t \| \nabla (C - \tilde{c})(\tau) \|^2 d\tau$$
$$\leq K_{18} \int_0^t \| (C - \tilde{c})(\tau) \|^2 d\tau + K_{19}\{h^{2r} + h^{2s-2}\} ,$$

where K_{18} depends, in particular, upon K^*. Then applying Gronwall's lemma to (3.13), we obtain

$$(3.14) \quad \| C - \tilde{c} \|_{L^\infty(J;L^2)} + \| \nabla (C - \tilde{c}) \|_{L^2(J;L^2)} \leq K_{20}\{h^r + h^{s-1}\} ,$$

where we note that

$$(3.15) \quad K_{20} \leq K_{21} \exp\{K_{22}K^{*2}\} .$$

To complete our argument, we must show that for h sufficiently small (3.6) holds. We use Lemma 3.1, (2.2), (2.11), (3.14), and (3.15) to see that

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$$\|\nabla P\|_{L^\infty(J;L^\infty)} \leq \|\nabla(P-\tilde{p})\|_{L^\infty(J;L^\infty)} + \|\nabla\tilde{p}\|_{L^\infty(J;L^\infty)}$$

$$\leq K_5 + K_0 h^{-1}\|\nabla(P-\tilde{p})\|_{L^\infty(J;L^2)}$$

(3.16)

$$\leq K_5 + K_0 h^{-1} K_6 \{\|c-\tilde{c}\|_{L^\infty(J;L^2)} + \|\tilde{c}-c\|_{L^\infty(J;L^2)}\}$$

$$\leq K_5 + K_0 K_6 K_{21}\exp\{K_{22}K^{*2}\}\{h^{r-1} + h^{s-2}\} + K_0 K_2 K_4 K_6 h^{r-1} .$$

Then clearly if h is taken sufficiently small, and if $r \geq 2$, and $s \geq 3$; then

(3.17)
$$\|\nabla P\|_{L^\infty(J;L^\infty)} \leq 2K_5 \leq K^* .$$

We see that, from (3.5), in order to get the best rates of convergence, one should use a space of piecewise polynomials of degree one greater for N_h that for M_h. For example, the use of piecewise linear polynomials $(r = 2)$ for M_h and piecewise quadratic polynomials $(r = 3)$ for N_h, would yield a rate of convergence of the form $O(h^2)$ for $c - \tilde{c}$.

We next combine the results of Lemma 3.1 and Theorem 3.2 with Lemmas 2.1 and 2.2 and the triangle inequality to obtain the following result.

Theorem 3.3. There exists a constant K_{23} such that

(3.18) $h\|\nabla(P - p)\|_{L^\infty(J;L^2)} + \|c - c\|_{L^\infty(J;L^2)} + h\|\nabla(c - c)\|_{L^2(J;L^2)} \leq K_{23}(h^r + h^{s-1})$.

Some mathematical models for miscible displacements which are currently being used by oil companies make the assumption that the coefficient b in (1.2) does not depend upon ∇p, but only upon x and c. Making this assumption, we obtain optimal order convergence rates in the L^2 norms using test spaces composed of piecewise polynomials of equal degrees. These results involve deriving a sharper estimate than obtained in (3.10).

Theorem 3.4. Assume $b = b(x,c)$ in (1.2) is independent of ∇p. There exists a constant $K_{24} = K_{24}(\lambda, b_*, \phi_*, K_0, K_1, i \leq 6; K^*, M)$ such that

(3.19) $h\|\nabla(P - p)\|_{L^\infty(J;L^2)} + \|c - c\|_{L^\infty(J;L^2)} + h\|\nabla(c - c)\|_{L^2(J;L^2)} \leq K_{24}(h^r + h^s + h^{r+s-3})$.

If either $r \geq 3$ or $s \geq 3$ for $r = 2$ and h is taken sufficiently small, then

$$(3.20) \qquad \| \nabla P \|_{L^\infty(J;L^\infty)} \leq 2K_5 \leq K^*$$

and a K_{24} can be chosen which is independent of K^*.

Proof. Most of the estimates will follow as in the proof of Theorem 3.2. Let $\zeta = C - \tilde{c}$. The estimate (3.9) can be replaced by

$$(3.21) \qquad \left| \int_0^t ([b(c) - b(C)]\nabla\tilde{c}, \nabla\zeta)d\tau \right| \leq K_{25} \int_0^t (\|c - \tilde{c}\| + \|\zeta\|)\|\nabla\zeta\|d\tau$$

$$\leq \frac{b_*}{8} \int_0^t \|\nabla\zeta(\tau)\|^2 d\tau + K_{25} \int_0^t \|\zeta(\tau)\|^2 d\tau + K_{26}h^{2r} .$$

In order to obtain a sharper estimate for (3.10), we rewrite

$$([u(c,\nabla p) - u(C,\nabla P)] \cdot \nabla\tilde{c}, \zeta) = ([u(c,\nabla p) - u(c,\nabla\tilde{p})] \cdot \nabla c, \zeta)$$

$$(3.22) \qquad + ([u(c,\nabla p) - u(c,\nabla\tilde{p})] \cdot \nabla(\tilde{c} - c), \zeta)$$

$$+ ([u(c,\nabla\tilde{p}) - u(C,\nabla P)] \cdot \nabla\tilde{c}, \zeta) = T_1 + T_2 + T_3 .$$

From Section 1, we note that for $u = u(c,q)$, $\dfrac{\partial u_i}{\partial q_i} = \dfrac{\partial u_i}{\partial q_i}(c)$ is independent of q, $i = 1,2$. Integrating the first term on the right-hand side of (3.22) by parts, we have

$$T_1 = -\left(p - \tilde{p}, \frac{\partial}{\partial x_1}\left[\frac{\partial u_1}{\partial q_1}(c) \frac{\partial c}{\partial x_1} \zeta\right] + \frac{\partial}{\partial x_2}\left[\frac{\partial u_2}{\partial q_2}(c) \frac{\partial c}{\partial x_2} \zeta\right] \right)$$

$$(3.23)$$

$$+ \int_{\partial\Omega} (p - \tilde{p})\zeta\left[\frac{\partial u_1}{\partial q_1}(c) \frac{\partial c}{\partial x_1} \nu_1 + \frac{\partial u_2}{\partial q_2}(c) \frac{\partial c}{\partial x_2} \nu_2\right]ds .$$

Thus, from the assumed smoothness of u and c, we have

$$(3.24) \quad |T_1| \leq K_{27}\|p - \tilde{p}\| \, \|\zeta\|_1 + \left| \int_{\partial\Omega} (p - \tilde{p})\zeta\left[\frac{\partial u_1}{\partial q_1}(c) \frac{\partial c}{\partial x_1} \nu_1 + \frac{\partial u_2}{\partial q_2}(c) \frac{\partial c}{\partial x_2} \nu_2\right]ds \right| .$$

We now estimate the above boundary term by considering the following Neumann problem:

$$(3.25) \quad \begin{array}{ll} a) & -\nabla \cdot a(x,c)\nabla\psi + \psi = 0, \qquad x \in \Omega , \\[2mm] b) & a(x,c) \dfrac{\partial\psi}{\partial\nu} = \Gamma , \qquad x \in \partial\Omega , \end{array}$$

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where

(3.26)
$$\Gamma = \zeta\left[\frac{\partial u_1}{\partial q_1}(c)\,\frac{\partial c}{\partial x_1}\,\nu_1 + \frac{\partial u_2}{\partial q_2}(c)\,\frac{\partial c}{\partial x_2}\,\nu_2\right]$$

$$\equiv \zeta G_1 .$$

By our regularity assumptions (S) and (R), we have $\psi \in H^2(\Omega)$ and

(3.27)
$$\|\psi\|_2 \le K\|\Gamma\|_{H^{1/2}(\partial\Omega)} .$$

We note that since $c \in L^\infty(J;H^{2+\epsilon}(\Omega))$ and $\frac{\partial u_i}{\partial q_i} = a(c)$ is uniformly bounded, $G_1 \in L^\infty(J;H^{1/2+\epsilon}(\partial\Omega))$ and we have by Lemma 2.2 of [1],

(3.28)
$$\|\Gamma\|_{H^{1/2}(\partial\Omega)} \le K\|G_1\|_{L^\infty(J;H^{1/2+\epsilon}(\partial\Omega))}\|\zeta\|_{H^{1/2}(\partial\Omega)}$$

$$\le K\|\zeta\|_1 .$$

Multiplying (3.25.a) by $p - \tilde{p}$ and integrating by parts, we see that

$$(a(x,c)\nabla\psi,\nabla(p - \tilde{p})) + (\psi, p - \tilde{p}) - \int_{\partial\Omega}\Gamma(p - \tilde{p})ds = 0 .$$

Using (2.6), we note that

(3.29)
$$\int_{\partial\Omega}\Gamma(p - \tilde{p})ds = (a(x,c)\nabla(\psi - \psi^*),\nabla(p - \tilde{p})) + (\psi, p - \tilde{p}), \quad \psi^* \in N_h .$$

Hence, by (2.1), Lemma 2.1, (3.27), (3.28), and (3.29), we have

(3.30)
$$\left|\int_{\partial\Omega}\Gamma(p-\tilde{p})ds\right| \le K_{28}\{h\|\nabla(p-\tilde{p})\| + \|p-\tilde{p}\|\}\|\psi\|_2 \le K_{29}\{h\|\nabla(p-\tilde{p})\| + \|p-\tilde{p}\|\}\|\Gamma\|_{H^{1/2}(\partial\Omega)}$$

$$\le K_{30}\{h\|\nabla(p-\tilde{p})\| + \|p-\tilde{p}\|\}\|\zeta\|_1 \le \frac{b_*}{8}\|\zeta\|_1^2 + K_{31}h^{2s} .$$

Combining (3.24), (3.30), and Lemma 2.1, we obtain

(3.31)
$$|T_1| \le \frac{b_*}{8}\|\zeta\|_1^2 + K_{32}\|\zeta\|^2 + K_{33}h^{2s} .$$

Using Lemmas 2.2 and 3.1 we note that the third term on the right of (3.22) can be bounded as follows

(3.32)
$$|T_3| \le K_{34}\|\nabla\tilde{c}\|_{L^\infty(J;L^\infty)}\{\|\nabla(\tilde{p} - P)\| + K^*\|c - C\|\}\|\zeta\| \le K_{35}\|\zeta\|^2 + K_{36}h^{2r} .$$

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In order to bound the second term on the right side of (3.22), we use (2.2) and
Lemmas 2.1 and 2.2. We obtain

$$(3.33) \quad |T_2| \leq K_{37} \|\nabla(p-\tilde{p})\|_{L^\infty(J;L^2)} \|\nabla(\tilde{c}-c)\| \, \|\zeta\|_{L^\infty} \leq K_{38} h^{s-2} \|\nabla(\tilde{c}-c)\| \, \|\zeta\|$$

$$\leq K_{39} h^{s-2} h^{r-1} \|\zeta\| \leq K_{40} \|\zeta\|^2 + K_{41} h^{2r+2s-6} \, .$$

Using the estimates (3.8), (3.11), (3.21), (3.22), (3.31), (3.32), and (3.33) in (3.7),
we obtain

$$(3.34) \quad \frac{\phi_*}{2} \|\zeta(t)\|^2 + \frac{3b_*}{4} \int_0^t \|\nabla\zeta(\tau)\|^2 d\tau \leq \frac{b_*}{4} \int_0^t \|\zeta(\tau)\|_1^2 d\tau + K_{42} \int_0^t \|\zeta(\tau)\|^2 d\tau$$

$$+ K_{43} \{h^{2r} + h^{2s} + h^{2r+2s-6}\} \, .$$

We can next add $\dfrac{3b_*}{4} \displaystyle\int_0^t \|\zeta(\tau)\|^2 d\tau$ to both sides of (3.34) and use the definition of
the H^1 norm to hide the first term on the right of (3.34) on the left side of the
resulting inequality. Then an application of Gronwall's lemma yields

$$(3.35) \quad \|\zeta\|_{L^\infty(J;L^2)} + \|\nabla\zeta\|_{L^2(J;L^2)} \leq K_{44} \{h^r + h^s + h^{r+s-3}\} \, .$$

The rest of the theorem follows as in Theorem 3.2 and Theorem 3.3 with the minor
modification of using (3.35) instead of (3.14) in (3.16) and the assumption that
$r + s \geq 5$.

4. A Priori Error Estimates for the Discrete Time Approximations

In this section, we develop a priori bounds for the errors $c^n - z^n$ and $\nabla(p^n - W^n)$ at discrete time levels t^n where z^n and W^n are defined in (2.15)-(2.16). The results obtained will be similar to the corresponding estimates for the continuous case from Section 3. We assume that S, Q, and R and the restrictions on $\{M_h\}$ and $\{N_h\}$ of Section 2 hold.

Theorem 4.1. Let z^0 be determined such that

$$a) \quad \|z^0 - c_0\| \leq K_{45} h^r$$

(4.0) and

$$b) \quad -\hat{\epsilon} \leq z^0 \leq 1 + \hat{\epsilon} .$$

There exists a constant $K_{46} = K_{46}(\lambda, b_\star, \phi_\star, M; K_i, i \leq 6)$ and constants $t_0 > 0$ and $h_0 > 0$ such that if $\Delta t \leq t_0$ and $h \leq h_0$,

(4.1)
$$\sup_{t^n} \|z - c\|^2 + h^2 \sum_{n=0}^{N-1} \|\nabla(z - c)^n\|^2 \Delta t + h^2 \sup_{t^n} \|\nabla(W - p)\|^2$$
$$\leq K_{46} \{(\Delta t)^2 + h^{2r} + h^{2(s-1)}\} .$$

Proof. Assume that z^n satisfies the induction hypothesis.

(4.2)
$$-\hat{\epsilon} \leq z^n \leq 1 + \hat{\epsilon}$$

for $n = 0, 1, \ldots, \ell - 1$. Let $\zeta^n = z^n - \tilde{c}^n$ and $\eta^n = W^n - \tilde{p}^n$. Subtracting (2.6) from (2.16) we have, for $n = 0, 1, \ldots, N - 1$,

(4.3)
$$(a(z^{n+1}) \nabla \eta^{n+1}, \nabla y) = ([a(c^{n+1}) - a(z^{n+1})] \nabla \tilde{p}^{n+1}, \nabla y)$$
$$+ ([a(z^{n+1}) \gamma(z^{n+1}) - a(c^{n+1}) \gamma(c^{n+1})] \nabla z, \nabla y), \qquad y \in N_h .$$

As in Lemma 3.1, we let $y = \eta^{n+1} \in N_h$, use (4.2) and (Q) to obtain

(4.4)
$$\|\nabla \eta^{n+1}\| \leq K_{47} \{\|\zeta^{n+1}\| + h^r\}, \qquad n = 0, 1, \ldots, N - 1 .$$

Next, subtract (2.9) from (2.15) and use the notation from Section 2 to obtain, for $n = 0, 1, \ldots, N - 1$, and $j = 0$ or 1,

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$$(\phi d_t \zeta^n, \chi) + (b(z^n, \nabla w^n) \nabla \zeta^{n+1}, \nabla \chi) = (\phi[\frac{\partial c}{\partial t}(t^{n+1}) - d_t \tilde{c}^n], \chi)$$

(4.5)
$$- \lambda((c - \tilde{c})^{n+1}, \chi) + ([b(c^{n+1}, \nabla p^{n+1}) - b(z^n, \nabla w^n)] \nabla \tilde{c}^{n+1}, \nabla \chi)$$

$$+ ([u(c^{n+1}, \nabla p^{n+1}) - u(z^n, \nabla w^n)] \cdot \nabla \tilde{c}^{n+j}, \chi)$$

$$- (u(z^n, \nabla w^n) \cdot \nabla \zeta^{n+j}, \chi), \qquad \chi \in M_h .$$

For simplicity, let

(4.6)
$$\|f\|_*^2 = (\phi f, f) .$$

Now letting $\chi = \zeta^{n+1} \in M_h$ and using (4.6), we estimate the left side of (4.5) by

(4.7)
$$\frac{1}{\Delta t}(\phi(\zeta^{n+1} - \zeta^n), \zeta^{n+1}) + (b(z^n, \nabla w^n) \nabla \zeta^{n+1}, \nabla \zeta^{n+1})$$

$$\geq \frac{1}{2\Delta t} \{\|\zeta^{n+1}\|_*^2 - \|\zeta^n\|_*^2\} + b_* \|\nabla \zeta^{n+1}\|^2 .$$

We next use Lemmas 2.1 and 2.2, (2.11), and (4.4) to see that

$$|([b(c^{n+1}, \nabla p^{n+1}) - b(z^n, \nabla w^n)] \nabla \tilde{c}^{n+1}, \nabla \zeta^{n+1})|$$

$$= |([b(c^{n+1}, \nabla p^{n+1}) - b(c^n, \nabla p^{n+1}) + b(c^n, \nabla p^{n+1}) - b(z^n, \nabla p^{n+1})$$

$$+ b(z^n, \nabla p^{n+1}) - b(z^n, \nabla p^n) + b(z^n, \nabla p^n) - b(z^n, \nabla w^n)] \nabla \tilde{c}^{n+1}, \nabla \zeta^{n+1})|$$

(4.8)
$$\leq \kappa_{50}\{\Delta t \|d_t c^n\| + \|(c - \tilde{c})^n\| + \|\zeta^n\| + \Delta t \|d_t \nabla p^n\| + \|\nabla(p - \tilde{p})^n\|$$

$$+ \|\nabla \eta^n\|\} \cdot \|\nabla \zeta^{n+1}\|$$

$$\leq \kappa_{51} \|\zeta^n\|^2 + \frac{b_*}{4} \|\nabla \zeta^{n+1}\|^2 + \kappa_{52}(\Delta t)^2 \{\|d_t c^n\|^2 + \|d_t \nabla p^n\|^2\} + \kappa_{53}\{h^{2r} + h^{2(s-1)}\} .$$

In a similar fashion, we obtain

$$|([u(c^{n+1}, \nabla p^{n+1}) - u(z^n, \nabla w^n)] \cdot \nabla \tilde{c}^{n+j}, \zeta^{n+1})|$$

(4.9)
$$\leq \frac{\phi_*}{16} \|\zeta^{n+1}\|^2 + \kappa_{54} \|\zeta^n\|^2 + \kappa_{55}(\Delta t)^2 \{\|d_t c^n\|^2 + \|d_t \nabla p^n\|^2\}$$

$$+ \kappa_{56}\{h^{2r} + h^{2(s-1)}\} .$$

Next, from (2.3 f), we note that

(4.10)
$$|-(u(z^n, \nabla w^n) \cdot \nabla \zeta^{n+j}, \zeta^{n+1})| \leq \kappa_{57}\{\|\nabla w^n\|_{L^\infty(\Omega)} + \kappa_{58}\}^2 \|\zeta^{n+1}\|^2 + \frac{b_*}{4} \|\nabla \zeta^{n+j}\|^2 .$$

We then estimate the first two terms on the right side of (4.5) using Lemma 2.2 as follows:

$$\left| (\phi[\frac{\partial c}{\partial t}(t^{n+1}) - d_t c^n + d_t c^n - d_t \tilde{c}^n], \zeta^{n+1}) - \lambda((c - \tilde{c})^{n+1}, \zeta^{n+1}) \right|$$

$$(4.11) \qquad \leq \frac{\phi_*}{16} \|\zeta^{n+1}\|^2 + \kappa_{59}\{\|(c - \tilde{c})^{n+1}\|^2 + \|d_t(c - \tilde{c})^n\|^2\} + \kappa_{60}\sigma_n^2$$

$$\leq \frac{\phi_*}{16} \|\zeta^{n+1}\|^2 + \kappa_{61}h^{2r} + \kappa_{60}\sigma_n^2 ,$$

where

$$(4.12) \qquad \sigma_n^2 = \left(\int_{t^n}^{t^{n+1}} \left\| \frac{\partial^2 c}{\partial t^2}(\cdot, s) \right\| ds \right)^2 \leq \Delta t \int_{t^n}^{t^{n+1}} \left\| \frac{\partial^2 c}{\partial t^2}(\cdot, s) \right\|^2 ds$$

by Schwarz's inequality. We note that

$$(4.13) \qquad \Delta t \sum_{n=0}^{N-1} \sigma_n^2 \leq (\Delta t)^2 \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \left\| \frac{\partial^2 c}{\partial t^2}(\cdot, s) \right\|^2 ds \leq (\Delta t)^2 \left\| \frac{\partial^2 c}{\partial t^2} \right\|_{L^2(J;L^2)}^2 .$$

We next multiply each of the estimates (4.7)-(4.11) by Δt and sum on n, $n = 0, \ldots, \ell - 1$, to obtain from (4.5)

$$\frac{1}{2}(\phi_* \|\zeta^\ell\|^2 - \phi^* \|\zeta^0\|^2) + b_* \sum_{n=0}^{\ell-1} \|\nabla \zeta^{n+1}\|^2 \Delta t$$

$$\leq \sum_{n=0}^{\ell-1} \frac{1}{2}\{\|\zeta^{n+1}\|_*^2 - \|\zeta^n\|_*^2\} + b_* \sum_{n=0}^{\ell-1} \|\nabla \zeta^{n+1}\|^2 \Delta t$$

$$(4.14)$$

$$\leq \kappa_{62} \sum_{n=0}^{\ell-1} \|\zeta^n\|^2 \Delta t + \frac{b_*}{2} \sum_{n=0}^{\ell-1} \|\nabla \zeta^{n+1}\|^2 \Delta t + \Delta t \frac{\phi_*}{8} \|\zeta^\ell\|^2$$

$$+ \kappa_{57} \sum_{n=0}^{\ell-1} \{\|\nabla w^n\|_{L^\infty(\Omega)} + \kappa_{58}\}^2 \|\zeta^{n+1}\|^2 \Delta t + \kappa_{63}\{(\Delta t)^2 + h^{2r} + h^{2(s-1)}\} .$$

We see that an application of Gronwall's lemma would complete our argument if we did not have the next to the last term on the right of (4.14). We shall use an induction argument to treat this term. As an induction hypothesis assume that for h sufficiently small

$$(4.15) \qquad \|\nabla w^n\|_{L^\infty(\Omega)} \leq 2\kappa_5, \quad \text{for} \quad n = 0, 1, \ldots, \ell - 1 .$$

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Using (2.2), Lemma 2.2, (2.11), (4.0), and (4.4) as in (3.16) we see that (4.15) clearly

holds for $\ell = 1$. Using (4.15) we see that if

(4.16)
$$\Delta t \leq \phi_* (4K_{57}\{2K_5 + K_{59}\}^2)^{-1} \equiv t_0 .$$

then we have

(4.17) $\quad \|\zeta^\ell\|^2 + b_* \sum_{n=0}^{\ell-1} \|\nabla \zeta^{n+1}\|^2 \Delta t \leq K_{64} \sum_{n=0}^{\ell-1} \|\zeta^n\|^2 \Delta t + K_{65}\{(\Delta t)^2 + h^{2r} + h^{2(s-1)}\} .$

Then an application of Gronwall's lemma gives us an estimate which, with (2.2), Lemma 2.2

(2.11), and (4.4) shows as in (3.16), that the induction hypotheses (4.2) and (4.15)

holds for $n = \ell$. Finally an application of the triangle inequality coupled with Lemmas

2.1 and 2.2 and (4.4) yields the desired result.

Just as in Section 3, if we make the assumption that $b = b(x,c)$ in (1.2) is

independent of ∇p, we can obtain optimal order L^2 rates of convergence. The proof

will follow by combining the techniques of Theorem 3.4 with those of Theorem 4.1. We

obtain the following result.

Theorem 4.2. Let the assumptions of Theorem 4.1 hold. Assume $b = b(x,c)$ in

(1.2) is independent of ∇p. There exist constants K_{66}, t_0, and h_0 such that, if

$\Delta t \leq t_0$ and $h \leq h_0$,

(4.18)
$$\sup_{t^n} \|z - c\|^2 + h^2 \sum_{n=0}^{N-1} \|\nabla(z - c)^n\|^2 \Delta t + h^2 \sup_{t^n} \|\nabla(w - p)\|^2$$

$$\leq K_{66}\{(\Delta t)^2 + h^{2r} + h^{2s} + h^{2r+2s-6}\} .$$

As we mentioned at the end of Section 2, if we replace (2.15) by (2.17) and extrap-

olate the coefficients, we can replace the $(\Delta t)^2$ in (4.17) and (4.18) by $(\Delta t)^4$ and

get discretization errors in time which are $O((\Delta t)^2)$. In order to do this we must

determine a starting procedure to obtain the approximation at time $t^1 = \Delta t$ since

two levels must be known to determine the next level with this method. A predictor-

corrector version of (2.15) will suffice as a starting procedure of sufficient accuracy.

Since the proof techniques are similar to those presented in Theorem 4.1 and are

similar to those presented in [1], [2], [5], [8] and [10], we shall not present them here. We require slightly more smoothness on c and p for these results.

Theorem 4.3. If we have the added smoothness on c and p that

$$(4.19) \qquad \left\| \frac{\partial^3 c}{\partial t^3} \right\|_{L^2(J;L^2)} + \left\| \frac{\partial^2 \nabla p}{\partial t^2} \right\|_{L^2(J;L^2)} \leq K_{67}$$

and if z^0 and z^1 are determined to satisfy

$$(4.20) \qquad \text{a)} \quad \| z^0 - c_0 \| + \| z^1 - c(t^1) \| \leq K_{68} h^r,$$

and

$$\text{b)} \quad -\hat{\epsilon} \leq z^n \leq 1 + \hat{\epsilon}, \qquad n = 0,1 ,$$

then there exist constants K_{69}, t_0, and h_0 such that, if $\Delta t \leq t_0$ and $h \leq h_0$,

$$(4.21) \qquad \sup_{t^n} \| \tilde{z} - c \|^2 + h^2 \sum_{n=0}^{N-1} \| \nabla (\tilde{z} - c)^n \|^2 \Delta t + h^2 \sup_{t^n} \| \nabla (W - p) \|^2$$

$$\leq K_{69} \{ (\Delta t)^4 + h^{2r} + h^{2s-2} \} .$$

Finally, in the case that $b = b(x,c)$, we can use the techniques of Theorem 4.2 to obtain the following result for $\tilde{z}$ defined in (2.17).

Theorem 4.4. Under the assumptions of Theorem 4.3, there exist constants K_{70}, t_0, and h_0 such that, if $\Delta t \leq t_0$ and $h \leq h_0$, we have

$$\sup_{t^n} \| \tilde{z} - c \|^2 + h^2 \sum_{n=0}^{N-1} \| \nabla (\tilde{z} - c)^n \|^2 \Delta t + h^2 \sup_{t^n} \| \nabla (W - p) \|^2$$

$$\leq K_{70} \{ (\Delta t)^4 + h^{2r} + h^{2s} \} .$$

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A priori error estimates for Galerkin methods for numerical approximation

of the coupled quasilinear system for $c - c(x,t)$ and $p - p(x,t)$ given by

$$\nabla \cdot [a(x,c)\{\nabla p - \gamma(x,c)\nabla z\}] = 0 ,$$

$$\nabla \cdot [b(x,c,\nabla p)\nabla c] - u(x,c,\nabla p)\cdot\nabla c = \phi(x)\frac{\partial c}{\partial t} ,$$

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20.   ABSTRACT - Cont'd.

for $x \in \Omega$, $t \in (0,T]$, and appropriate Neumann boundary and initial conditions
are considered.   Equations of this type arise in models for the miscible
displacement of one incompressible fluid by another in a porous medium.
Estimates for both continuous time and fully-discrete time Galerkin methods
are presented.